Financial Formulas C

Financial Calculator Basics

The financial calculator freed mortgage brokers, appraisers, and other real estate professionals from dependence on printed tables of compound interest rates and sinking fund factors. Previously, calculating the present value of an investment or the growth of compound interest involved relatively simple algebraic calculations, assuming the appropriate interest rate or factor could be found in the financial tables. A typical collection of compound interest rate tables, however, would only include quarter-point increments, so calculating the future value of a loan made at an 8.55% annual rate forced analysts to make an educated guess based on data in the 8½% and 8¾% tables or to use the financial formula $S^n = (1 + i)^n$. In contrast, an analyst who uses a financial calculator can choose any rate or holding period in calculations related to the *six functions of one*, which is a significant advance in flexibility and accuracy over working with financial tables.

Hewlett Packard's HP-12C financial calculator, in particular, is notable for its simplicity and staying power. It has remained relatively unchanged in form and function since its introduction in the early 1980s. The newer HP-12C Platinum calculator includes the ability to toggle between algebraic and Reverse Polish Notation systems and has more memory spaces, but otherwise the more recent version performs the same mathematical and statistical functions.

Reverse Polish Notation (RPN) refers to a mathematical order of operations developed to facilitate chain calculations. In short, numerals are entered before mathematical operators. For example, the sequence of keystrokes used to perform the arithmetic function 5 + 2 on a typical calculator and the output display at each step would be

Keystrokes	Display
5	5
$\boxed{+}$	5 +
2	2
$\boxed{=}$	7

On an RPN calculator, the sequence of keystrokes and output would be

Keystrokes	Display
5	5
ENTER	5.00
2	2
+	7.00

Compare two sample sequences of keystrokes for a more complex series of calculations, say, for the arithmetic expression (5 + 2) × (12 – 4):

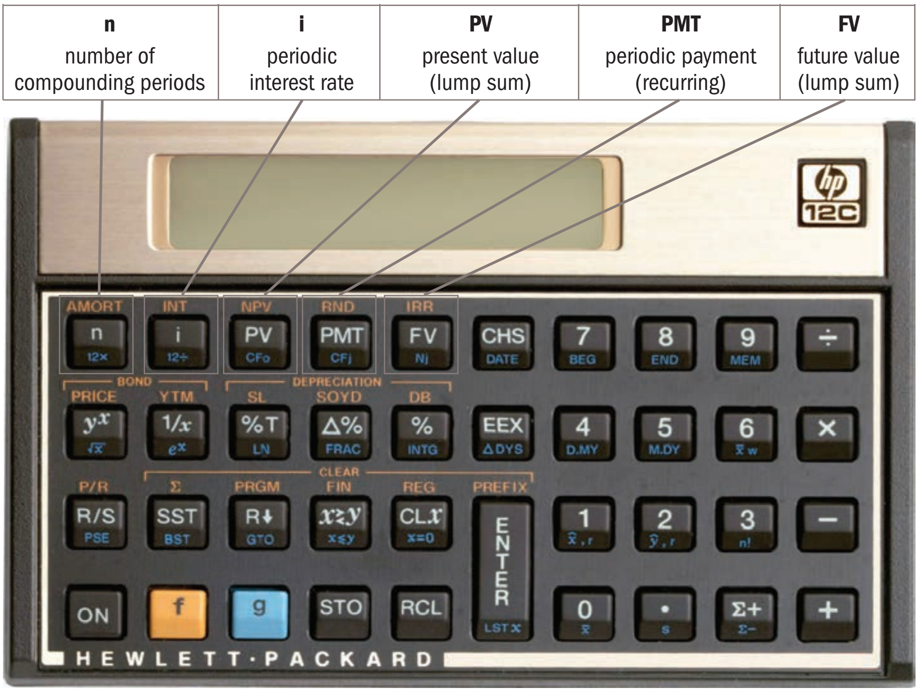
Algebraic Keystrokes	RPN Keystrokes
5	5
+	ENTER
2	2
=	+
STO	12
12	ENTER
–	4
4	–
=	X
X	
RCL	
=	

Using the algebraic keystrokes, the interim result of the first parenthetical expression must be stored in the calculator's memory or jotted down on a piece of scrap paper and reentered at the appropriate point in the chain of calculations. An HP-12C, however, automatically stores the displayed figures in continuous storages areas known as *stack registers*. (The function and operation of the storage registers in a financial calculator are explained in detail in the owner's manual.) Note that the route to the answer has no effect on the answer itself.

Performing simple mathematical calculations using Reverse Polish Notation can take some getting used to, but the benefits gained in performing financial analysis on a financial calculator are well worth it. Not only does the calculation using RPN require fewer keystrokes in the example above but, for experienced real estate professionals, the keystrokes become more intuitive than similar calculations would be on a calculator using a different operating system.

Common uses of a financial calculator by real estate professionals include computing mortgage payments (i.e., amortization calculations), discounting cash flows, and, of particular importance to residential appraisers, making cash equivalency adjustments for comparable sales with atypical financing or concessions.

Most of the variables related to the time value of money can be calculated with a few keystrokes. These variables have a prominent place across the top of the HP-12C keypad:



As an example, consider a fully amortized $150,000 mortgage loan with a 30-year term at a fixed rate of 6.35%. Calculating the monthly payment is straightforward using the HP-12C. The known mortgage terms are input, and the calculator then solves for the variable requested:

Calculation	Keystrokes	Display
Enter the term of the loan (and convert to number of months)	30 [g] [n]	360.00
Enter the annual interest rate (and convert to the effective monthly interest rate)	6.35 [g] [i]	0.53
Enter the loan amount	150,000 [CHS] [PV]	-150,000.00
Calculate the monthly payment	[PMT]	933.35

An analyst can then compare the result, $933.35, to market rent for a comparable property in the market area and answer the client's question of whether to buy or rent. A sophisticated user can extend the chain of calculations to determine the equity build-up at any point in the term of the loan, change the interest rate at a certain point in the loan term to model refinancing, and make other changes to address many other situations.

Note that other sequences of keystrokes could be used to solve the same problem with the HP-12C. For example, in the sequence of keystrokes above, the annual interest rate could be entered before the term of the loan without affecting the results generated in solving for the payment.

A financial calculator facilitates the calculation of annual, biannual, quarterly, or even daily payments, and allows for quick comparisons of various loan terms. As an example, consider the same loan with annual, end-of-year payments:

Calculation	Keystrokes	Display
Clear the registers	f CLx	0.00
Enter the term of the loan (in years)	30 n	30.00
Enter the annual interest rate	6.35 i	6.35
Enter the loan amount	150,000 CHS PV	1150,000.00
Calculate the monthly payment	PMT	11,308.53

The annual payment, $11,308.53, works out to be a larger amount than the total of a dozen monthly payments (12 × $933.35 = $11,200.20) in the same year even though the same nominal interest rate, 6.35%, is used in both calculations. When annual payments are made, the principal is only reduced at the end of each year, so the amount of interest paid in that year is calculated based on the entire principal balance at the beginning of the year. In contrast, the principal balance (from which the portion of the monthly payment attributable to interest is calculated) drops incrementally in a monthly amortization schedule.[1] As a result, over the course of the year the portion of the $933.35 monthly payment attributable to the lender's return on the investment (i.e., the interest paid on the loan) becomes smaller. The difference between one annual payment and the total of 12 monthly payments on a loan reflects the time value of money. The cash flows received throughout the year have a higher future value than the income the lender receives at the end of the year because those funds can be reinvested and continue to grow.

More sophisticated analyses with a financial calculator include calculating the payment schedule, yield, and price or present value of balloon mortgages, graduated payment mortgages, wraparound loans, construction loans, and other types of loans. In addition to mortgage analysis, a financial calculator can be used in the application of the income capitalization approach to calculate lease payments, the future value of an investment, discount rates, or other information needed for discounted cash flow analyses.

1. As an illustration of the decrease in interest paid as a portion of the constant monthly payment, consult typical amortization schedules readily available online from lenders, mortgage brokers, and others.

Basic Formulas

Symbols

I = income
R = capitalization rate
V = value
M = mortgage ratio
DCR = debt coverage ratio
F = capitalization factor (multiplier)
GIM = gross income multiplier
$EGIM$ = effective gross income multiplier
NIR = net income ratio

Subscript:

O = overall property
M = mortgage
E = equity
L = land
B = building
LF = leased fee
LH = leasehold

Basic Income/Cap Rate/Value Formulas

$I = R \times V$

$R = I/V$

$V = I/R$

Basic Value/Income/Factor Formulas

$V = I \times F$

$I = V/F$

$F = V/I$

Adaptations for Mortgage/Equity Components

Band of investment (using ratios)

$$R_O = M \times R_M + [(1 - M) \times R_E]$$

$$R_E = \frac{R_O - (M \times R_M)}{1 - M}$$

Equity residual

$$V_O = \frac{I_O - (V_M \times R_M)}{R_E} + V_M$$

$$R_E = \frac{I_O - (V_M \times R_M)}{V_E}$$

Mortgage residual

$$V_O = \frac{I_O - (V_E \times R_E)}{R_M} + V_E$$

Debt coverage ratio

$$R_O = DCR \times M \times R_M$$

$$DCR = \frac{R_O}{M \times R_M}$$

$$M = \frac{R_O}{DCR \times R_M}$$

Cap Rate/Factor Relationships

$R = 1/F$

$R_O = NIR/GIM$

$R_O = NIR/EGIM$

Note: *NIR* may relate to scheduled gross or effective gross income; care should be taken to ensure consistency.

Adaptations for Land/Building Components

Land residual

$$V_O = \frac{I_O - (V_B \times R_B)}{R_L} + V_B$$

$$R_L = \frac{I_O - (V_B \times R_B)}{V_L}$$

Building residual

$$V_O = \frac{I_O - (V_L \times R_L)}{R_B} + V_L$$

$$R_B = \frac{I_O - (V_L \times R_L)}{V_B}$$

Discounted Cash Flow Analysis Formulas

Symbols

PV = present value
CF = cash flow
Y = yield rate
R = capitalization rate
Δ = change
a = annualizer
$1/S_{\overline{n|}}$ = sinking fund factor
$1/n$ = 1/projection period
CR = compound rate of change
V = value

Subscript:
n = projection periods
O = overall property
I = income

Discounted Cash Flows/Present Value (DCF/*PV*)

$$PV = \frac{CF_1}{(1+Y)} + \frac{CF_2}{(1+Y)^2} + \frac{CF_3}{(1+Y)^3} + \dots + \frac{CF_n}{(1+Y)^n}$$

Basic Capitalization Rate/Yield Rate/Value Change Formulas

$$R = Y - \Delta a$$
$$Y = R + \Delta a$$
$$\Delta a = Y - R$$
$$\Delta = \frac{Y - R}{a}$$

Adaptations for Common Income/Value Patterns

Pattern	Premise	Cap Rate (*R*)	Yield Rate (*Y*)	Value Ch ange (Δ)
Perpetuity	$\Delta = 0$	$R = Y$	$Y = R$	
Level annuity*	$a = 1/S_{\overline{n|}}$	$R = Y - \Delta\ 1/S_{\overline{n|}}$	$Y = R + \Delta\ 1/S_{\overline{n|}}$	$\Delta = \frac{Y-R}{1/S_{\overline{n|}}}$
Straight-line change	$a = 1/n$	$R = Y - \Delta\ 1/n$	$Y = R + \Delta\ 1/n$	$\Delta = \frac{Y-R}{1/n}$
Exponential change	$\Delta a = CR$	$R_O = Y_O - CR$	$Y_O = R_O + CR$	$\Delta = (1+CR)^n - 1$

* Inwood premise: $1/S_{\overline{n|}}$ at Y rate; Hoskold premise: $1/S_{\overline{n|}}$ at safe rate

Straight-Line Change* in Income	Straight-Line Change* in Value	Compound Rate of Change
$\$\Delta_I = V \times \Delta\ 1/n \times Y$	$\$\Delta\ 1/n = \$\Delta_I / Y$	$CR = \sqrt[n]{FV/PV} - 1$
$\Delta_I = \frac{Y \times \Delta\ 1/n}{Y - \Delta\ 1/n}$	$\Delta\ 1/n = \frac{Y \times \Delta_I}{Y + \Delta_I}$	$CR = Y_O - R_O$

* In these formulas Δ_I is the ratio of one year's change in income to the first year's income.

Six Functions of One

The following formulas may be used to convert the annual constant (R_M) for a monthly payment loan to the corresponding monthly functions.

Function for Monthly Frequency	Formula
Amount of one	$S^n = \dfrac{R_M}{R_M - I}$
Amount of one per month	$S^n = \dfrac{12}{R_M - I}$
Sinking fund factor	$1/S_{\overline{n}\rceil} = \dfrac{R_M - I}{12}$
Present value of one	$1/S^n = \dfrac{R_M - I}{R_M}$
Present value of one per month	$a_{\overline{n}\rceil} = \dfrac{12}{R_M}$
Partial payment	$1/a_{\overline{n}\rceil} = \dfrac{R_M}{12}$

In these formulas, I = nominal interest rate.

Present Value of Level Annuities

The Inwood Premise

The Inwood premise applies to income that is an ordinary level annuity. It holds that the present value of a stream of income is based on a single discount rate. Each installment of income is discounted with a single discount rate, and the total discounted values of the installments are accumulated to obtain the present value of the income stream. The present value of a series of $1 payments can be found in compound interest tables for a given rate and a given period of time. It is assumed that the income will be sufficient to return all investment capital to the investor and to pay the specified return on the investment.

In most mortgages, the amount of interest declines gradually over the holding period and is calculated as a specified percentage of the unrecaptured capital. Any excess over the required interest payment is considered a return of capital and reduces the amount of capital remaining in the investment. Because the installments are always the same amount, the principal portion of the payments increases by the same amounts that the interest portion of the payments decreases. It is also valid, but not customary, to see the interest payments as constant, always amounting to the specified return on the original investment, with any excess over the required, fixed-interest payments credited to a hypothetical sinking fund that grows with interest at the same rate to repay the original investment.

An Inwood capitalization rate can be constructed by adding the interest rate to a sinking fund factor ($1/S_{\overline{n}\rceil}$) that is based on the same interest rate and duration as the income stream. The resulting capitalization rate is simply the reciprocal of the ordinary level annuity (present value of one per period) factor found in financial tables. Thus, the Inwood premise is consistent with the use of compound interest tables to calculate the present value of the income stream.

The Inwood premise applies only to a level stream of income. Therefore, the present value of any expected reversion or other benefit not included in the income stream must be added to obtain the total present value of the investment. For example, assume that the net operting income (I_O) of a property is $10,000 per year for five years. What is the value of the property assuming an overall yield rate (Y_O) of 10% under the Inwood premise?

Solution 1

Apply the PV of 1 per period (ordinary level annuity) factor to the I_O:

$$3.79079 \times \$10{,}000 = \$37{,}908 \text{ (rounded)}$$

Solution 2

The general yield capitalization formula can also be used for a level income with a percentage change in value:

$$R_O = Y_O - \Delta_O \, 1/S_{\overline{n}|}$$

Because there is no reversion, the property will lose 100% of its value. Thus, Δ_O is -1.0 and the yield capitalization formula becomes

$$R_O = Y_O + 1/S_{\overline{n}|}$$

With appropriate inputs, this equation represents the Inwood premise. By substituting the data given in the example, R_O can be solved for as follows:

$$R_O = 0.10 + 0.163797$$
$$R_O = 0.263797$$

The value of the property may be estimated using the basic valuation formula:

$$\begin{aligned} V_O &= I_O / R_O \\ &= \$10{,}000/0.263797 \\ &= \$37{,}908 \text{ (rounded)} \end{aligned}$$

Note that the sinking fund factor ($1/S_{\overline{n}|}$) is based on a 10% discount rate, which implies that a portion of the I_O could be reinvested at 10% to replace the investment. It can be said that Y_O represents the return on capital and $1/S_{\overline{n}|}$ represents the return of capital.

The Inwood premise assumes a constant rate of return on capital each year with the return of capital being reinvested in a sinking fund at the same yield rate as Y_O. The amount accumulated in this sinking fund can be used to replace the asset at the end of its economic life. Using the assumptions applied in the preceding example, the net operating income for the first year may be allocated as follows:

I_O	\$10,000.00
Return on Capital (10% of \$37,908)	– \$3,790.80
Return of Capital	\$6,209.20

If the return of capital (\$6,209.20) is placed in a sinking fund earning 10%, the fund will accumulate to \$37,908 over five years. The sinking fund accumulation factor (future value of one per period), $S_{\overline{n}|}$, is applied to the return of capital:

$$6.1051 \times \$6{,}209.20 = \$37{,}908$$

This is the exact amount required to replace the asset.

The Hoskold Premise

The Hoskold premise differs from the Inwood premise in that it employs two separate interest rates:

- A speculative rate, representing a fair rate of return on capital commensurate with the risks involved

- A safe rate for a sinking fund, designed to return all the invested capital to the investor in a lump sum at the termination of the investment

In contrast to the Inwood premise, the Hoskold premise assumes that the portion of net operating income needed to recover or replace capital (the return of capital) is reinvested at a "safe rate"—e.g., the prevailing rate for insured savings accounts or government bonds—which is lower than the "speculative" yield rate (Y_O) used to value the other portion of I_O. Like the Inwood premise, the Hoskold technique was designed to be applied when the asset value of the investment decreases to zero over the holding period. However, Hoskold assumed that funds would have to be set aside at a lower, safe rate to replace the asset at the end of the holding period. Hoskold suggested that this technique might be appropriate for valuing wasting assets such as a mine where the value is reduced to zero as minerals are extracted; thus funds have to be set aside to invest in a new mine once the minerals are totally depleted—i.e., the reversion equals zero.

Using the same net operating income, yield, and term set forth in the previous example, assume that a portion of I_O has to be set aside at a 5% safe rate to allow for the recovery of capital at the end of every five-year period. All other assumptions remain the same. This problem may be solved with the same yield capitalization formula applied in the Inwood calculation, but the sinking fund factor ($1/S_{\overline{n}|}$) is based on the safe rate of 5% rather than the yield rate of 10%. Thus, the overall rate is calculated as follows:

$$
\begin{aligned}
R_O &= Y_O + 1/S_{\overline{n}|} \\
&= 0.10 + 0.180975 \\
&= 0.280975
\end{aligned}
$$

Because the sinking fund factor ($1/S_{\overline{n}|}$) is calculated at a 5% rate rather than the 10% rate, the capitalization rate is higher and the value is lower. The value is calculated as:

$$
\begin{aligned}
V_O &= I_O / R_O \\
&= \$10{,}000/0.280975 \\
&= \$35{,}590 \text{ (rounded)}
\end{aligned}
$$

The lower value is a result of setting aside the portion of I_O earning 5% to allow for the recovery of capital ($35,590) at the end of five years. The income allocation for the first year can be shown as follows:

I_O	$10,000
Return on Capital (10% of $35,590)	– 3,559
Return of Capital	$6,441

To find the future value of $6,441 at 5% for five years, apply the sinking fund accumulation factor (future value of one per period), $S_{\overline{n}|}$, to the return of capital:

$$5.525631 \times \$6{,}441 = \$35{,}590$$

The result is the exact amount required to recover the capital invested.

Present Value of Increasing/Decreasing Annuities

Straight-Line Changes

To obtain the present value of an annuity that has a starting income of *d* at the end of the first period and *increases h dollars* per period for *n* periods:

$$PV = (d + h\,n)\,a_{\overline{n}|} - \frac{h\,(n - a_{\overline{n}|})}{i}$$

To obtain the present value of an annuity that has a starting income of *d* at the end of the first period and *decreases h dollars* per period for *n* periods, simply make *h* negative in the formula.

Exponential-Curve (Constant-Ratio) Changes

To obtain the present value of an annuity that starts at $1 at the end of the first period and increases each period thereafter at the rate *x* for *n* periods:

$$PV = \frac{1 - (1 + x)^n/(1 + i)^n}{i - x}$$

where *i* is the periodic discount rate and *x* is the ratio between the increase in income for any period and the income for the previous period.

To obtain the present value of an annuity that starts at $1 at the end of the first period and decreases each period thereafter at rate *x*, simply make *x* negative in the formula.

Rates of Return

Symbols

PV = present value
NPV = net present value
CF = cash flow
i = discount rate (in NPV formula)
n = projection period
IRR = internal rate of return
PI = profitability index
$MIRR$ = modified internal rate of return
$FVCFj$ = future value of a series of cash flows
i = reinvestment rate (in $MIRR$ formula)

Subscript:
0 = at time zero
1 = end of 1st period
2 = end of 2nd period
3 = end of 3rd period
n = end period of series

Net Present Value (*NPV*)

$$NPV = CF_0 + \frac{CF_1}{(1 + i)} + \frac{CF_2}{(1 + i)^2} + \frac{CF_3}{(1 + i)^3} + \dots + \frac{CF_n}{(1 + i)^n}$$

Internal Rate of Return (*IRR*)

Where $NPV = 0$; $IRR = i$

Profitability Index (*PI*)

$$PI = PV/CF_0$$

Modified Internal Rate of Return (*MIRR*)

$$MIRR = \sqrt[n]{\frac{FVCFj}{CF_0}} - 1$$

$$MIRR = \sqrt[n]{\frac{CF_1 (1 + i)^{n-1} + CF_2 (1 + i)^{n-2} + CF_3 (1 + i)^{n-3} + \ldots + CF_n}{CF_0}} - 1$$

Note: In these formulas individual *CF*s may be positive or negative for *PV* and *NPV* solutions; however, CF_0 is treated as a positive value for *PI* and *MIRR* solutions.

Mortgage Interests

Mortgage investments have a great effect on real property value and equity yield rates. Because yield is a significant consideration in the lender's decision to invest in a mortgage interest in real estate, the lender's yield must be understood and often calculated. In the absence of points and any participation or accrual feature, the lender's yield equals the interest rate.

Mortgage information used to value income-producing properties may include

1. The monthly or periodic payments and annual debt service on a level-payment, fully amortized loan
2. The accompanying partial payment factors and annual constants (R_M)
3. The balance outstanding (B) on an amortized loan at any time before it is fully amortized, expressed as a dollar amount or a percentage of the original loan amount
4. The percentage or proportion of the principal amount paid off before full amortization (P)

Mortgage Components

Periodic (Monthly) Payment

The monthly payment factor for a fully amortized, monthly payment loan with equal payments is the direct reduction loan factor, or *monthly constant*, for the loan, given the interest rate and amortization term. Thus, the monthly payment factor for a 30-year, fully amortized, level monthly payment loan at 15.5% interest is 0.013045. This number can be obtained from a direct reduction loan table or by solving for the monthly payment (*PMT*) on a financial calculator, given the number of periods (n), the interest rate (i), and the principal loan amount.

If the loan had an initial principal amount of $160,000, the monthly payment required to amortize the principal over 30 years and provide interest at the nominal rate of 15.5% on the outstanding balance each month would be

$$\$160{,}000 \times 0.013045 = \$2{,}087.20$$

Annual Debt Service and Loan Constant

Cash flows are typically converted to an annual basis for real property valuation, so it is useful to calculate the amount of annual debt service as well as the monthly payments. For the 30-year, fully amortized, level monthly payment loan of $160,000 at a 15.5% interest rate, the annual debt service is

$$\$2{,}087.20 \times 12 = \$25{,}046.40$$

The annual loan constant is simply the ratio of annual debt service to the loan principal. (The annual loan constant, often called the *mortgage constant*, describes a rate although it is actually the annual debt service per dollar of mortgage loan outstanding, which may be expressed as a dollar amount.) The annual loan constant is expressed as R_M to signify that it is a capitalization rate for the loan or debt portion of the real property investment. For the loan mentioned, the annual loan constant can be calculated as follows:

$$\begin{aligned} R_M &= \frac{\text{Annual Debt Service}}{\text{Loan Principal}} \\ &= \frac{\$25{,}046.40}{\$160{,}000.00} \\ &= 0.156540 \end{aligned}$$

The annual loan constant can also be obtained when the amount of the loan principal is not known. In this case, the monthly payment factor is simply multiplied by 12.

$$\begin{aligned} R_M &= \text{Monthly Payment Factor} \times 12 \\ &= 0.013045 \times 12 \\ &= 0.156540 \end{aligned}$$

Although these figures are rounded to the nearest cent, in actual practice most loan constants are rounded up to make sure that the loan will be repaid during the stated amortization period.

Outstanding Balance

Properties are frequently sold, or loans may be refinanced, before the loan on the property is fully amortized. Furthermore, loans often mature before the completion of loan amortization. In these cases, there is an outstanding balance or balloon payment due on the note. From the lender's point of view, this is the loan or debt reversion to the lender.

The outstanding balance (B) on any level-payment, amortized loan is the present value of the debt service over the remaining amortization period discounted at the interest rate. Thus, at the end of 10 years, the balance for the 30-year note discussed above would be the present value of 20 years of remaining payments. The balance is calculated by multiplying the monthly payment by the present value of one per period factor (monthly) for 20 years at the interest rate. The balloon payment, or future value, may be calculated.

$$\begin{aligned} B &= \$2{,}087.20 \times 73.861752 \\ &= \$154{,}164.25 \end{aligned}$$

Similarly, the outstanding balance at the end of 18 years would be equal to the monthly payment times the present value of one per period factor (monthly) for 12 years at the interest rate.

$$\begin{aligned} B &= \$2{,}087.20 \times 65.222881 \\ &= \$136{,}133.20 \end{aligned}$$

The outstanding balance on a loan can also be expressed as a percentage of the original principal. This is useful, and sometimes necessary, if dollar amounts are not given or are unavailable. For a 10-year projection with 20 years remaining on the note, the outstanding balance is

$$B = \frac{\$154{,}164.25}{\$160{,}000.00}$$
$$= 0.963527$$

For an 18-year projection with 12 years remaining on the note, the balance is

$$B = \frac{\$136{,}133.20}{\$160{,}000.00}$$
$$= 0.850833$$

A percentage balance can also be calculated as the ratio of the present value of one per period factor for the remaining term of the loan at the specified interest rate divided by the present value of one per period factor for the full term of the loan at the interest rate. This can be expressed as

$$B = \frac{PV\ 1/P \text{ Remaining Term}}{PV\ 1/P \text{ Full Term}}$$

In the case of the 30-year, 15.5% loan, the balance for a 10-year projection with 20 years remaining is calculated as

$$B = \frac{73.861752}{76.656729}$$
$$= 0.963539$$

For an 18-year projection with 12 years remaining, the balance would be

$$B = \frac{65.222881}{76.656729}$$
$$= 0.850844$$

These results are similar to those obtained using dollar amounts.

Percentage of Loan Paid Off

It is often necessary to calculate the percentage of the loan paid off before full amortization over the projection period, especially in Ellwood mortgage-equity analysis. The percentage of the loan paid off is expressed as *P* and is most readily calculated as the complement of *B*.

$$P = 1 - B$$

For the 30-year note, *P* is calculated as follows:

$$P_{10} = 1 - 0.963539$$
$$= 0.036461$$
$$P_{18} = 1 - 0.850844$$
$$= 0.149156$$

The percentage of the loan paid off prior to full amortization over the projection period (*P*) can also be calculated directly. There are many different procedures for this operation and they are not all presented here. Calculator users are advised to consult their manuals on the AMORT function.

The simplest, most direct procedure is to calculate P as the ratio of the sinking fund factor for the full term (monthly) divided by the sinking fund factor for the projection period (monthly).

$$P = \frac{1/S_{\overline{n}|}}{1/S_{\overline{n}|_p}}$$

For the 30-year monthly payment note at 15.5%, the calculations are

$$P_{10} = \frac{0.000129}{0.003524} = 0.036606 \qquad P_{18} = \frac{0.000129}{0.000862} = 0.149652$$

Any differences are due to rounding.

Lender's Yield

To illustrate how the lender's yield on a mortgage loan investment is calculated, consider a mortgage loan with the following characteristics.

Loan amount	$100,000
Interest rate	13.5%
Term	25 years
Payment	Monthly
Balance in five years	$96,544
Points	3
Other costs	Borrower to pay all other costs

If the mortgage runs full term, the yield can be obtained using a calculator.

$n = 300$
$PMT = \$1,165.65$
$PV = \$97,000$ ($100,000 less 3 points, or $3,000)*
$i = 13.97\%$

* Each point is equal to 1% of the loan amount: $100,000 × 0.01 = $1,000.

The lender's yield is greater than the nominal interest rate because of the points paid by the borrower. In effect, the lender only loaned $97,000 ($100,000 – $3,000) but receives a stream of debt service payments based on $100,000. If the mortgage is paid off in five years, the lender's yield is calculated with these figures.

$n = 60$
$PMT = \$1,165.65$
$PV = \$97,000$
$FV = \$96,544$
$i = 14.36\%$

If there were no points in either of these examples, the yield to the lender would be 13.5% in each case. Points or any other monetary payments that reduce the lender's investment are important considerations in calculating the lender's yield. The lender's yield may be supplemented through the syndication process.

In some depressed markets, lenders may find that the property securing the loan has declined in value to the point that the loan balance exceeds the property's value. In this case, there is no longer any equity interest in the property, and the value of the loan may often be calculated based on the actual cash flows to the property rather than the cash flows projected when the loan contract was obtained. To do otherwise would be to estimate the value of the mortgage interest as greater than the value of the property.

Mortgage/Equity Formulas

Symbols

r = basic capitalization rate
Y = yield rate
M = mortgage ratio
C = mortgage coefficient
P = ratio paid off—mortgage
$1/S_{\overline{n}|}$ = sinking fund factor
R = capitalization rate
$S_{\overline{n}|}$ = future value of one per period
Δ = change
J = J factor (changing income)
n = projection period
I_O = net operating income
B = mortgage balance
I = nominal interest rate

Subscript:
E = equity
M = mortgage
P = projection
O = overall property
I = income
1 = 1st mortgage
2 = 2nd mortgage

Basic Capitalization Rate (r)

$$r = Y_E - M\,C$$

$$r = Y_E - (M_1\,C_1 + M_2\,C_2)$$

$$C = Y_E + P\;1/S_{\overline{n}|} - R_M$$

$$P = \frac{R_M - I}{R_{M_p} - I}$$

$$P = 1/S_{\overline{n}|} \times S_{\overline{n}|_p}$$

Capitalization Rates (R)

Level income

$$R = Y_E - M\,C - \Delta\;1/S_{\overline{n}|}$$

$$R = r - \Delta\;1/S_{\overline{n}|}$$

***J*-factor changing income**

$$R_O = \frac{Y_E - M\,C - \Delta_O\;1/S_{\overline{n}|}}{1 + \Delta_I\,J}$$

$$R_O = \frac{r - \Delta_O\;1/S_{\overline{n}|}}{1 + \Delta_I\,J}$$

Required Change in Value (Δ)

Level income

$$\Delta = \frac{r - R}{1/S_{\overline{n}|}}$$

$$\Delta = \frac{Y_E - M\,C - R}{1/S_{\overline{n}|}}$$

***J*-factor changing income**

$$\Delta_O = \frac{r - R_O\,(1 + \Delta_I\,J)}{1/S_{\overline{n}|}}$$

$${}^*\Delta_O = \frac{r - R_O}{R_O\,J + 1/S_{\overline{n}|}}$$

Note: For multiple mortgage situations, insert M and C for each mortgage.

* This formula assumes value and income change at the same ratio.

Equity Yield (Y_E)

Level income

$$Y_E = R_E + \Delta_E\ 1/S_{\overline{n}|}$$

***J*-factor changing income**

$$Y_E = R_E + \Delta_E\ 1/S_{\overline{n}|} + \left(\frac{R_O\ \Delta_I}{1 - M}\right) J$$

Change in equity

$$\Delta_E = \frac{\Delta_O + M\ P}{1 - M}$$

$$\Delta_E = \frac{V_O\ (1 + \Delta_O) - B - V_E}{V_E}$$

Assumed mortgage situation

Level income

$$V_O = \frac{I_O + B\ C}{Y_E - \Delta_O\ 1/S_{\overline{n}|}}$$

***J*-factor changing income**

$$V_O = \frac{I_O\ (1 + \Delta_I\ J) + B\ C}{Y_E - \Delta_O\ 1/S_{\overline{n}|}}$$

Mortgage/Equity Without Algebra Format

Loan Ratio × Annual Constant	=	________
Equity Ratio × Equity Yield Rate	= +	________
Loan Ratio × Paid Off Loan Ratio × SFF	= −	________
Basic Rate (*r*)	=	________
+ Dep or − App × SFF	= ±	________
Capitalization Rate (*R*)	=	________

Note: SFF is sinking fund factor at equity yield rate for projection period. Dep/App is the change in value from depreciation or appreciation during the projection period.

Mortgage-Equity Analysis

L. W. Ellwood was the first to organize, develop, and promulgate the use of mortgage-equity analysis in yield capitalization for real property valuation. He theorized that mortgage money plays a major role in determining real property prices and values. Ellwood saw real property investments as a combination of two components—debt and equity—and held that the return requirements of both components must be satisfied through income, reversion, or a combination of the two. Thus, Ellwood developed an approach for estimating property value that made explicit assumptions as to what a mortgage lender and an equity investor would expect from the property.

In general, mortgage-equity analysis involves estimating the value of a property on the basis of both mortgage and equity return requirements. The value of the equity interest in the property is found by discounting the equity dividends available to the equity investor. The equity yield rate (Y_E) is used as the discount rate. The total value of the property is equal to the present value of the equity position plus the value of the mortgage. This is true whether the value is found using discounted cash flow analysis or yield capitalization formulas that have been developed for mortgage-equity analysis.

Applications

Mortgage-equity analysis can facilitate the valuation process in many ways. It may be used

1. To compose overall rates
2. To analyze and test the capitalization rates obtained with other capitalization techniques
3. As an investment analysis tool to test the values indicated by the sales comparison and cost approaches
4. To analyze a capitalization rate graphically

Given a set of assumptions concerning the net operating income, mortgage (amount, rate, and term), reversion (rate of appreciation or depreciation), equity yield rate, and projection period, mortgage-equity analysis may be employed to estimate the present value of the equity and to arrive at the total property value. The following example illustrates a general approach to mortgage-equity analysis.

Given:

Annual I_0 (level)	$25,000
Projection period	10 years
Loan amount	$168,000
Loan terms*	
Interest rate	9%
Amortization term (monthly payments)	25 years
Estimated reversion	$201,600
Equity yield rate	15%

* Contract terms are at current market rates.

Using these assumptions, cash flow to the equity investor can be projected as follows:

Annual Cash Flow from Operations—Years 1–10	
Annual Net Operating Income	$25,000
Annual Debt Service	− 16,918
Equity Dividend	$8,082
Cash Flow from Reversion—Year 10	
Estimated Resale Price	$201,600
Mortgage Balance	− 139,002
Cash Flow from Reversion	$62,598

Using the present value factor for a 15% yield rate and a 10-year holding period, the present value of the cash flows to the equity investor may be calculated as follows:

Years	Cash Flow	Present Value Factor	Present Value
1–10	$8,082	5.018769*	$40,562
10	$62,598	0.247185†	+ 15,473
Present Value of Equity			$56,035

* Ordinary level annuity (present value of one per period) factor

† Reversion (present value of one) factor

The total property value can now be found by adding the present value of the equity to the present value of the loan.[2]

Present Value of the Equity	$56,035
Present Value of the Loan	+ 168,000
Total Value	$224,035

This example illustrates a fairly straightforward application of mortgage-equity analysis. The present value of the equity was easily calculated by discounting the dollar estimates of the cash flows. The assumptions in this example were simplified in several ways. First, the income was assumed to be level. In a more complex situation, income may be expected to change over the holding period. Second, the loan amount was specified in dollars.[3] If the loan amount were assumed to be based on a loan-to-value ratio, the dollar amount of the loan would depend on the property value being calculated. In such a case the cash flows to the equity investor could not be specified in dollars and discounted as they were in the example. Third, the resale price was specified in dollars.[4] Investors often assume that property values will change by a specified percentage amount over the holding period (see Chapter 24). Thus, the resale price depends on the property value being calculated. Finally, in the preceding example the total property value is greater than the loan amount. If the opposite were true, the value of the loan could not exceed the combined debt and equity interests in the property.

When either the loan amount or the resale price depends on the value of the property, the cash flows cannot be projected in dollar amounts and discounted. An alternative procedure must be used to solve for the present value. One such alternative is to use a yield capitalization formula that has been developed to solve this type of problem.[5] This is what L. W. Ellwood did when he developed the Ellwood equation, which is illustrated in the following section.

Mortgage-Equity Formula

The general mortgage-equity formula is:

$$R_O = \frac{Y_E - M\,(Y_E + P\ 1/S_{\overline{n}|} - R_M) - \Delta_O\ 1/S_{\overline{n}|}}{1 + \Delta_I\, J}$$

where:

R_O = overall capitalization rate
Y_E = equity yield rate
M = loan-to-value ratio
P = percentage of loan paid off
$1/S_{\overline{n}|}$ = sinking fund factor at the equity yield rate
R_M = mortgage capitalization rate or mortgage constant
Δ_O = change in total property value
Δ_I = total ratio change in income
J = J factor (This symbol is discussed later in this appendix.)

2. Because the loan is assumed to be at current market rates, the face amount of the loan is equal to the value of the loan to the lender.
3. This might be the case if the property were being valued subject to an existing loan. Such a situation is illustrated later in this appendix. Alternatively, the dollar amount may have resulted from a separate calculation of the maximum amount that could be borrowed to meet a minimum debt coverage ratio.
4. This situation might occur if there is a purchase option in a lease that the appraiser believes will be exercised. Alternatively, a dollar estimate may be the result of a separate estimate of the resale price calculated by applying a capitalization rate to the income at the end of the holding period.
5. For a discussion of this procedure, see Jeffrey D. Fisher, "Using Circular Reference in Spreadsheets to Estimate Value," *The Quarterly Byte*, vol. 5, no. 4 (Fourth Quarter 1989).

The part of the formula represented as $Y_E - M\,(Y_E + P\;1/S_{\overline{n|}} - R_M)$ can be referred to as the *basic capitalization rate* (r). It satisfies the lender's requirement and adjusts for amortization. It also satisfies the investor's equity yield requirement before any adjustment is made for income and value changes. Therefore, the basic rate starts with an investor's yield requirement and adjusts it to reflect the effect of financing. The resulting basic capitalization rate is a building block from which an overall capitalization rate can be developed with additional assumptions.

If level income and no change in property value are anticipated, the basic rate will be identical to the overall capitalization rate. The last part of the numerator, $\Delta_O\;1/S_{\overline{n|}}$, allows an appraiser to adjust the basic rate to reflect an expected change in overall property value. If the value change is positive, referred to as *property appreciation*, the overall capitalization rate is reduced to reflect this anticipated monetary benefit. If the change is negative—referred to as *depreciation*—the overall capitalization rate is increased.

Finally, the denominator, $1 + \Delta_I\,J$, accounts for any change in income. The J factor is always positive. Thus, if the change in income is positive, the denominator will be greater than one and the overall rate will be reduced. If the change in income is negative, the overall rate will be increased. For level-income applications, $\Delta = 0$, so the denominator is $1 + 0$, or 1.

Akerson Format

The mortgage-equity procedure developed by Charles B. Akerson substitutes an arithmetic format for the algebraic equation in the Ellwood formula.[6] This format is applicable to level-income situations; when modified with the J or K factor, it can also be applied to changing-income situations.

The Akerson format for level-income situations is

Loan Ratio × Annual Constant	=	________
Equity Ratio × Equity Yield Rate	= +	________
Loan Ratio × % Paid Off in Projection Period × $1/S_{\overline{n|}}$	= −	________
Basic Rate (r)	=	________
+ Dep or − App × $1/S_{\overline{n|}}$	= ±	________
Overall Capitalization Rate	=	________

where $1/S_{\overline{n|}}$ is the sinking fund factor at the equity yield rate for the projection period and *dep/app* denotes the change in value from property depreciation or appreciation during the projection period.

Level-Income Applications

Mortgage-equity analysis can be used to value real property investments with level income streams or variable income streams converted to level equivalents using overall capitalization rates and residual techniques.

Use of Overall Capitalization Rates

In the simplest application of the mortgage-equity formula and the Akerson format, a level income and a stable or changing overall property value are assumed. The fol-

6. The format was first presented by Charles B. Akerson in "Ellwood without Algebra," *The Appraisal Journal* (July 1970): 325-335.

lowing example illustrates the application of the mortgage-equity formula using an overall capitalization rate applied to a level flow of income.

I_O (level)	$25,000
Projection period	10 years
Loan terms	
Interest rate	9%
Amortization term (monthly payments)	25 years
Loan-to-value ratio	75%
Property value change	20% gain
Equity yield rate	15%

The overall rate is calculated as follows:

$$R_O = \frac{Y_E - M\,(Y_E + P\;1/S_{\overline{n}|} - R_M) - \Delta_O\;1/S_{\overline{n}|}}{1 + \Delta_I\,J}$$

$$R_O = \frac{0.15 - 0.75\,(0.15 + 0.1726 \times 0.04925 - 0.1007) - (0.20 \times 0.04925)}{1 + 0 \times J}$$

$$R_O = \frac{0.15 - 0.75\,(0.057801) - 0.009850}{1}$$

$$R_O = \frac{0.15 - 0.043350 - 0.009850}{1}$$

$$R_O = \frac{0.096800}{1}$$

$$R_O = 0.0968 \text{ (rounded)}$$

The capitalized value of the investment is $25,000/0.0968 = $258,264.

Using the same data and assumptions, an identical value can be derived by applying the Akerson format:

0.75×0.100704	=	0.075528
0.25×0.15	=	+ 0.037500
$-0.75 \times 0.172608 \times 0.049252$	=	− 0.006376
Basic Rate (r)	=	0.106652
0.20×-0.049252	=	− 0.009850
R_O	=	0.096802
The capitalized value is $25,000/0.0968	=	$258,264

The answer derived in this example is virtually the same as the answer that would be derived using DCF analysis. In fact, it is possible to check the answer found with the Ellwood formula by discounting the implied cash flows. This is true because the dollar amount of the loan and resale price are approximately the same in both examples. That is, the implied amount of the loan is 75% of $224,014, or approximately $168,000, and the implied resale price is 90% of $224,014, or approximately $201,600. It is important to realize, however, that this was not known until the problem was solved. The examples were designed to produce the same answer to demonstrate that both problems are based on the same concepts of discounted cash flow analysis.

Use of Residual Techniques

Land and building residual techniques can be applied with land and building capitalization rates based on mortgage-equity procedures. The general mortgage-equity formula or the Akerson format is applied to derive a basic rate, which is used to develop land and building capitalization rates.

For example, assume that a commercial property is expected to produce level annual income of $15,000 per year over a 10-year term. Mortgage financing is available at a 75% loan-to-value ratio, and monthly payments at 11% interest are made over an amortization term of 25 years. The land is currently valued at $65,000 and is forecast to have a value of $78,000 at the end of the projection period, indicating a 20% positive change in land value. The building is expected to have no value at the end of the projection period, and the equity yield rate is 15%.

The first step in valuing this property is to derive the basic rate (r) using the Ellwood Formula:

$$\begin{aligned} r &= Y_E - M\,(Y_E + P\ 1/S_{\overline{n}|} - R_M) \\ r &= 0.15 - 0.75\,(0.15 + 0.137678 \times 0.049252 - 0.117614) \\ &= 0.15 - 0.029375 \\ &= 0.120625 \end{aligned}$$

The Akerson format can also be used to derive the basic rate:

0.75 × 0.117614	=	0.088211
0.25 × 0.15	=	0.037500
0.75 × 0.137678 × 0.049252	= –	0.005086
Basic Capitalization Rate (r)	=	0.120625

Next, the land and building capitalization rates are calculated. To solve for the land capitalization rate, R_L, the calculations are

$$\begin{aligned} R_L &= r - \Delta_L\ 1/S_{\overline{n}|} \\ &= 0.120625 - (0.20 \times 0.049252) \\ &= 0.120625 - 0.009850 \\ &= 0.110775 \end{aligned}$$

The building capitalization rate, R_B, is calculated as follows:

$$\begin{aligned} R_B &= r - \Delta_B\ 1/S_{\overline{n}|} \\ &= 0.120625 - (-1.0 \times 0.049252) \\ &= 0.120625 + 0.049252 \\ &= 0.169877 \end{aligned}$$

These rates can be used to value the property with the building residual technique:

I_O		$15,000
Land Income		
($V_L \times R_L$) = $65,000 × 0.110775	–	7,200
Residual Income Attributable to Building		$7,800
Capitalized Value of Building		
($I_B \div R_B$) = $7,800/0.169877		$45,916
Plus Land Value		65,000
Indicated Property Value		$110,916

When the rates are used in the land residual technique, a similar property value is indicated:

I_O		\$15,000
Building Income		
$(V_B \times R_B) = \$46,000 \times 0.169877$	–	7,814
Residual Income Attributable to Land		\$7,186
Capitalized Value of Land		
$(I_L \div R_L) = \$7,186/0.110775$		\$64,870
Plus Building Value		46,000
Indicated Total Property Value		\$110,870

Changing-Income Applications

The general mortgage-equity formula can be applied to income streams that are forecast to change on a curvilinear or exponential-curve (constant-ratio) basis by using a *J* factor for curvilinear change or a *K* factor for constant-ratio change. The *J* factor, used in the stabilizer $(1 + \Delta_I J)$, may be obtained from precomputed tables or calculated with the *J*-factor formula.[7] The *K* factor, an income adjuster or stabilizer used to convert a changing income stream into its level equivalent, can be calculated with the *K*-factor formula.[8]

Use of the *J* Factor

The *J*-factor formula for curvilinear income reflects an income stream that changes from time zero in relation to a sinking fund accumulation curve. The formula is

$$J = 1/S_{\overline{n}|} \times \left(\frac{n}{1 - 1/(1+Y)^n} - \frac{1}{Y}\right)$$

where:

$1/S_{\overline{n}|}$ = sinking fund factor at equity yield rate

n = projection period

Y = equity yield rate

Consider the facts set forth in the level annuity example, but assume a 20% increase in income. Note that the *J* factor is applied to the income in the year prior to the first year of the holding period.

$$\begin{aligned}
R_O &= \frac{0.15 - 0.75\ (0.15 + 0.172608 \times 0.049252 - 0.100704) - (0.20 \times 0.049252)}{1 + (0.20 \times 0.3259)} \\
&= \frac{0.15 - 0.043348 - 0.009850}{1 + 0.0652} \\
&= \frac{0.096802}{1.0652} \\
&= 0.09088
\end{aligned}$$

The capitalized value is \$25,000/0.09088 = \$275,088.

7. Before the advent of financial calculators, present value and future value problems were solved using precomputed tables of compound interest factors. Although the tables are no longer used in everyday practice, they remain useful for checking results of calculations made with calculators and computers and in teaching the mathematics of finance. See James J. Mason, ed., comp., *American Institute of Real Estate Appraisers Financial Tables*, rev. ed. (Chicago: American Institute of Real Estate Appraisers, with tables computed by Financial Publishing Company, 1982), 461-473.
8. Charles B. Akerson, *Capitalization Theory and Techniques: Study Guide*, 3rd ed., ed. David C. Lennhoff (Chicago: Appraisal Institute, with tables computed by Financial Publishing Company, 2000), T-47 to T-52.

The net operating incomes for the projection period that are implied by the curvilinear *J*-factor premise are calculated in the following table.

Period	First Year Adjustment*		$S_{\overline{n}\rceil}$		Periodic Adjustment		Base I_0†		I_0
1	$246.26	×	1/1.000000	=	$246	+	$25,000	=	$25,246
2	$246.26	×	1/0.465116	=	$529	+	$25,000	=	$25,529
3	$246.26	×	1/0.287977	=	$855	+	$25,000	=	$25,855
4	$246.26	×	1/0.200265	=	$1,230	+	$25,000	=	$26,230
5	$246.26	×	1/0.148316	=	$1,660	+	$25,000	=	$26,660
6	$246.26	×	1/0.114237	=	$2,156	+	$25,000	=	$27,156
7	$246.26	×	1/0.090360	=	$2,725	+	$25,000	=	$27,725
8	$246.26	×	1/0.072850	=	$3,380	+	$25,000	=	$28,380
9	$246.26	×	1/0.059574	=	$4,134	+	$25,000	=	$29,134
10	$246.26	×	1/0.049252	=	$5,000	+	$25,000	=	$30,000

* This adjustment was derived by multiplying I_0 ($25,000) by the assumed increase in I_0 (20%); the resulting figure ($5,000) was then multiplied by the sinking fund factor for the anticipated 15% equity yield rate over the 10-year projection period ($1/S_{\overline{n}\rceil} = 0.049252$).

† The base I_0 is the income for the year prior to the beginning of the projection period.

Mathematical proof of the example is provided below.

Valuation of Equity

Period	Net Operating Income		Debt Service		Cash to Equity		*PVF* at 15%		*PV*
1	$25,246	–	$20,772	=	$4,474	×	0.869565	=	$3,890
2	$25,529	–	$20,772	=	$4,757	×	0.756144	=	$3,597
3	$25,855	–	$20,772	=	$5,083	×	0.657516	=	$3,342
4	$26,230	–	$20,772	=	$5,458	×	0.571753	=	$3,121
5	$26,660	–	$20,772	=	$5,888	×	0.497177	=	$2,927
6	$27,156	–	$20,772	=	$6,384	×	0.432328	=	$2,760
7	$27,725	–	$20,772	=	$6,953	×	0.375937	=	$2,614
8	$28,380	–	$20,772	=	$7,608	×	0.326902	=	$2,487
9	$29,134	–	$20,772	=	$8,362	×	0.284262	=	$2,377
10	$30,000	–	$20,772	=	$9,228	×	0.247185	=	$2,281
					$159,400*	×	0.247185	=	$39,401
Value of equity at 15%								=	$68,797
Check: $275,088 × 0.25 = $68,772									

* The reversion is calculated as follows:

Resale ($275,088 × 1.20)	=	$330,106
Loan Balance ($275,088 × 0.75)(1 – 0.1726)	=	– $170,706
Equity Proceeds	=	$159,400

Use of the *K* Factor

The *K*-factor formula, which is applied to income that changes on an exponential-curve (constant-ratio) basis, is expressed as

$$K = \frac{1 - (1 + C)^n / S^n}{(Y - C)\, a_{\overline{n}\rceil}}$$

where:

K = factor

C = constant-ratio change in income

S^n = future value factor

Y = equity yield rate

$a_{\overline{n}\rceil}$ = present value factor for ordinary level annuity

When the general mortgage-equity formula is used to derive an overall capitalization rate applicable to an income expected to change on a constant-ratio basis, K is substituted for the denominator $(1 + \Delta_I J)$. The following example is based on the same property used for the level-income and J-factor examples, but it assumes that net operating income will increase by 2% per year, on a compound basis. This property can be valued using the K factor in the mortgage-equity formula.

$$R_O = \frac{Y_E - M\,[Y_E + P\,1/S_{\overline{n}|} - R_M] - \Delta_O\,1/S_{\overline{n}|}}{K}$$

$$R_O = \frac{0.15 - 0.75\,(0.15 + 0.172608 \times 0.049252 - 0.100704) - (0.20 \times 0.049252)}{1.070877}$$

$$= 0.090395$$

The capitalized value of the investment is \$25,000/0.090395 = \$276,564.

Note that the indicated values based on the J-factor and K-factor premises are very close, i.e., \$275,088 and \$276,564. The indicated value based on a level-income assumption is much lower, i.e., \$238,264. This is because all of the yield has to occur on resale, not in increased income.

Based on the income data in Table C.1, J-factor and K-factor income patterns are plotted on the graph in Figure C.1. Both examples assume a 20% increase in overall property value. In the J-factor example, income is projected to increase by 20%. In the K-factor example, income is projected to increase at a constant ratio of 2% per year. Under the J-factor assumption, the value of the property is \$275,088 and the R_O is 9.088%. Under the K-factor assumption, the value of the property is \$276,569, and the R_O is 9.039%.

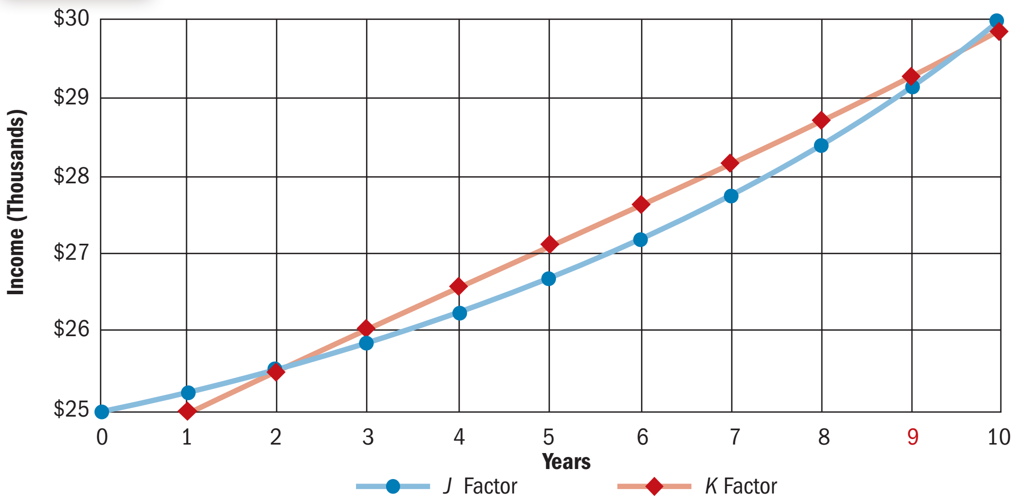
Solving for Equity Yield

Given an actual or proposed equity sale price and a forecast of equity benefits, an equity yield rate can be estimated. When level income is forecast, a formula is used. The calculations can be performed by iteration or with the financial functions of a calculator. When income is expected to change on a curvilinear basis or a constant-ratio basis, formulas must be used to solve for the yield. A calculator cannot be used to solve the problem conveniently, and the iteration technique is too time-consuming.

Table C.1 *J*-Factor Income Pattern and *K*-Factor Income Pattern

	J Factor	*K* Factor
Year 1	\$25,246	\$25,000
Year 2	\$25,529	\$25,500
Year 3	\$25,855	\$26,010
Year 4	\$26,230	\$26,530
Year 5	\$26,660	\$27,060
Year 6	\$27,156	\$27,602
Year 7	\$27,725	\$28,154
Year 8	\$28,380	\$28,717
Year 9	\$29,134	\$29,291
Year 10	\$30,000	\$29,877

Figure C.1 *J*-Factor and *K*-Factor Income Pattern Curves



Level-Income Example

Consider a property that is purchased for $250,000. The net operating income is forecast to remain level at $35,000 per year, and the buyer believes that property value will decline 15% over a five-year ownership period. The mortgage amount is $200,000 and monthly payments are at 10% interest with an amortization term of 20 years. The investment forecast is outlined below:

Purchase		Holding Period	
Sale Price	$250,000	I_O	$35,000
Mortgage	– 200,000	Debt Service	– 23,161*
Equity	$50,000	Equity Dividend	$11,839

Resale After 5 Years	
Sale Price	$212,500
Mortgage Balance	– 179,605†
Equity Reversion	$32,895
Original Equity	– 50,000
Equity Change	-$17,105

* $200,000 × 0.115803 mortgage constant

† Unamortized portion of $200,000 mortgage at end of 5-year projection period

$$R_E \text{ (Equity Capitalization Rate)} = \frac{\$11{,}839}{\$50{,}000} = 0.236780$$

$$\Delta_E \text{ (Equity Change)} = \frac{-\$17{,}105}{\$50{,}000} = -0.342100$$

The equity yield rate may now be computed through iteration or by using the formula and interpolation. Iteration is performed using the formula

$$Y_E = R_E + \Delta_E \, 1/S_{\overline{n|}}$$

Because the sinking fund factor for five years at the Y_E rate cannot be identified without knowing Y_E, a trial-and-error procedure must be used to develop Y_E. Without discounting, the 34.21% equity decline over the five-year holding period would subtract 6.84% each year from the equity capitalization rate of 23.67%. Consequently, Y_E will be less than 23.67% and more than 16.83% (23.67% – 6.84%).

The first computation is performed with a Y_E of 18%. When the correct equity yield rate is applied, the equation will balance.

Estimated Y_E	R_E	+	Δ_E	×	$1/S_{\overline{n}\rceil}$	=	Indicated Y_E
0.1800	0.2368	+	(-0.3425)	×	0.139778	=	0.1889
0.2000	0.2368	+	(-0.3425)	×	0.134380	=	0.1908
0.1900	0.2368	+	(-0.3425)	×	0.137050	=	0.1899

Therefore, $Y_E = 0.1900$, or 19.0%.

This procedure for computing Y_E is correct because Y_E is defined as the rate that makes the present value of the future equity benefits equal to the original equity. The future benefits in this case are the equity dividend of $11,839 per year for five years and the equity reversion of $32,895 at the end of the five-year period.

If Y_E is 19%, the present value of the two benefits can be computed.

$$\begin{aligned} \$11{,}839 \times 3.057635 &= \quad \$36{,}199 \\ \$32{,}895 \times 0.419049 &= + \ \underline{\quad 13{,}785} \\ & \qquad \$49{,}984 \end{aligned}$$

Thus, the equity yield rate has been proven to be 19.0%. Precision to 0.03% represents a level of accuracy in keeping with current practice and the normal requirements of the calculation. This example is based on level income, but the same procedure can be applied to changing income streams by incorporating *J* and *K* factors into the formula.

J-Factor Premise Example

Consider the information set forth in the previous example, but assume that income is expected to decline 15% according to the *J*-factor premise.

$R_O = \$35{,}000/\$250{,}000 = 0.14$ $\qquad M = \$200{,}000/\$250{,}000 = 0.80$

$$Y_E = R_E + \frac{\Delta_E}{S_{\overline{n}\rceil}} + \frac{R_O \Delta_I}{1 - M} J$$

Try 15%,

$$0.2368 + -0.3421 \times 0.1483 + \frac{0.14 \times -0.15}{0.2} \times 0.4861 = 0.135$$

Try 12%,

$$0.2368 + -0.3421 \times 0.1574 + \frac{0.14 \times -0.15}{0.2} \times 0.5077 = 0.130$$

Try 13%,

$$0.2368 + -0.3421 \times 0.1543 + \frac{0.14 \times -0.15}{0.2} \times 0.5004 = 0.131472$$

Therefore, $Y_E = 13.15\%$ (rounded).

K-Factor Premise Example

Consider the same information, but assume that income is expected to decrease at a compound rate of 3% per year, indicating a constant-ratio change in income.

$$Y_E = R_E + \Delta_E\ 1/S_{\overline{n|}} + \frac{R_O\,(K - 1)}{1 - M}$$

Try 13%,

$$0.2368 + \text{-}0.3421 \times 0.1543 + \frac{0.14 \times (0.9487 - 1)}{0.2} = 0.148$$

Try 15%,

$$0.2368 + \text{-}0.3421 \times 0.1483 + \frac{0.14 \times (0.9497 - 1)}{0.2} = 0.151$$

Therefore, $Y_E = 15.1\%$.

Rate Analysis

Rate analysis allows an appraiser to test the reasonableness of the value conclusions derived through the application of overall capitalization rates. Once an overall capitalization rate has been developed with mortgage-equity analysis or another technique, its reliability and consistency with market expectations of equity yield and value change can be tested using Ellwood graphic analysis.

To create a graph for rate analysis, an appraiser chooses equity yield rates that cover a realistic range of rates expected and demanded by investors. It is often wise to include a rate that is at the low end of the range of market acceptance as well as a rate at the high end of the range. For the analysis to be useful to the client, the range of yield rates chosen should be in line with investors' perceptions of the market.

In most real estate investments, there is no assurance that the investment can be liquidated at the convenience of the equity investor or on the terms dictated by the investor. For example, in the early 1990s most liquidity evaporated from the market. Moreover, in negotiating a purchase price, the prospects for profit within a plausible range of possibilities may be greater than the chance of achieving a specific equity yield rate, which cannot be determined until the property is resold. However, the appraiser's value judgments can easily be subjected to realistic tests. The appraiser should ask the following questions:

- What resale prices correspond to various yield levels?
- Can the property suffer some loss in value and still produce an acceptable profit?
- How sensitive is the equity yield rate to possible fluctuations in value?
- What percentage of the investor's return is derived from annual cash flows, and what percentage comes from the reversion? (Reversion is generally considered riskier.)
- What prospective equity yield rates can be inferred from the overall capitalization rates found in the marketplace?

Many of these questions focus on the relationship between the change in property value and the equity yield rate. The unknown variable in rate analysis is the change in property value (Δ_O). The formula for the required change in property value in a level-income application is

$$\Delta_O = \frac{r - R_O}{1/S_{\overline{n|}}}$$

Level-Income Example

Consider an investment that will generate stable income and has an overall capitalization rate of 10%. The purchase can be financed with a 75% loan at 10% interest amortized over 25 years with level monthly payments. If the investment is held for 10 years, what levels of depreciation or appreciation should be expected with equity yield rates of 9%, 12%, and 15%?

To solve this problem the appraiser must first find the basic rate (r) and the sinking fund factor for each equity yield rate. The Ellwood Tables[9] are the source of the following figures:

Y_E	r	$1/S_{\overline{n}\rceil}$
9%	0.096658	0.065820
12%	0.105185	0.056984
15%	0.113584	0.049252

When the difference between r and the overall rate (R_O) is divided by the corresponding sinking fund factor, the result is the expected change in property value. If r is greater than R_O, a value increase is indicated; if r is less than R_O, a value loss is indicated. Analysis of the 10% overall capitalization rate is shown below:

$$Y_E = \frac{r - R_O}{1/S_{\overline{n}\rceil}}$$

9%	-0.0508 (5.1% depreciation)
12%	0.0910 (9.1% appreciation)
15%	0.2758 (27.6% appreciation)

The formula produces answers consistent with the notion that a loss is negative and a gain is positive. In some texts the numerator in this formula is expressed as $R_O - r$. Use of this formula results in a change of sign—i.e., positive answers indicate depreciation and negative answers indicate appreciation.

J-Factor Premise

A similar analysis can be performed when income is presumed to change commensurately with value according to the *J*-factor premise. In this case, the expected change in overall property value is calculated by dividing ($r - R_O$) by ($R_O J + 1/S_{\overline{n}\rceil}$).

Graphic Rate Analysis

Various systems have been developed to employ mortgage-equity concepts in graphic rate analysis. The graphic analysis of capitalization rates is a helpful analytical tool used by practicing appraisers and investment analysts. Rate analysis in graphic or tabular form is particularly useful in interpreting market data. Although analyzing a market-oriented overall capitalization rate cannot reveal a property's eventual equity yield rate or resale price, the analysis can reveal combinations of Y_E and Δ_O implicit in the overall rate. Thus, an appraiser can use rate analysis to decide whether a particular combination of Y_E and Δ_O is consistent with market behavior.

The accompanying figures illustrate two types of graphic analysis. Figure C.2 shows Ellwood-style graphic analysis, with time on the horizontal axis and the percentage change in property value on the vertical axis. Figure C.3 illustrates another

9. L. W. Ellwood, *Ellwood Tables for Real Estate Appraising and Financing*, 4th ed. (Cambridge, Mass.: Ballinger Publishing Co., 1977).

Figure C.2 Ellwood-style Graphic Analysis

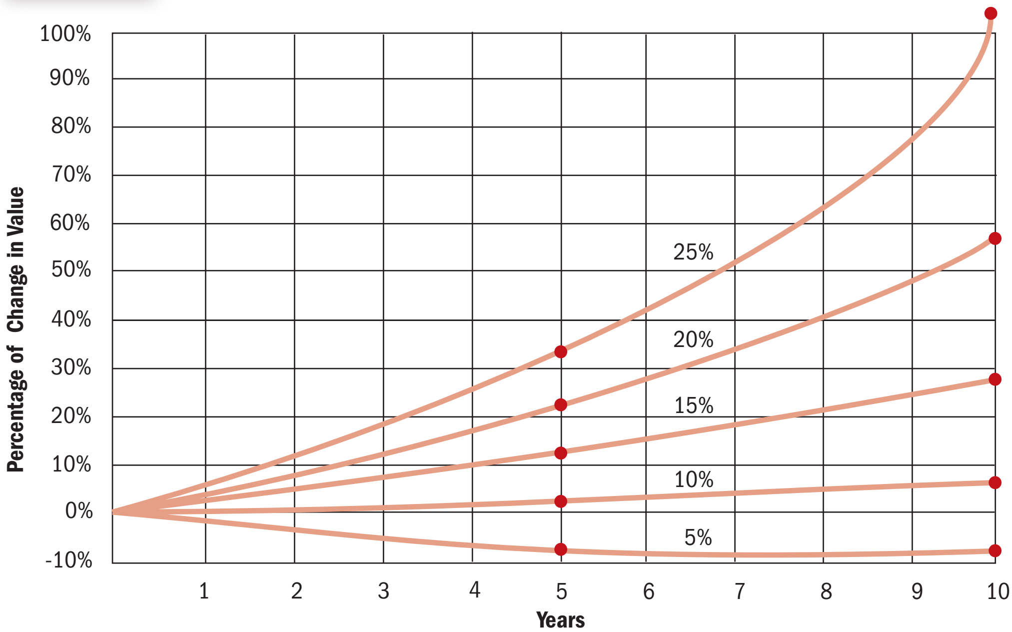
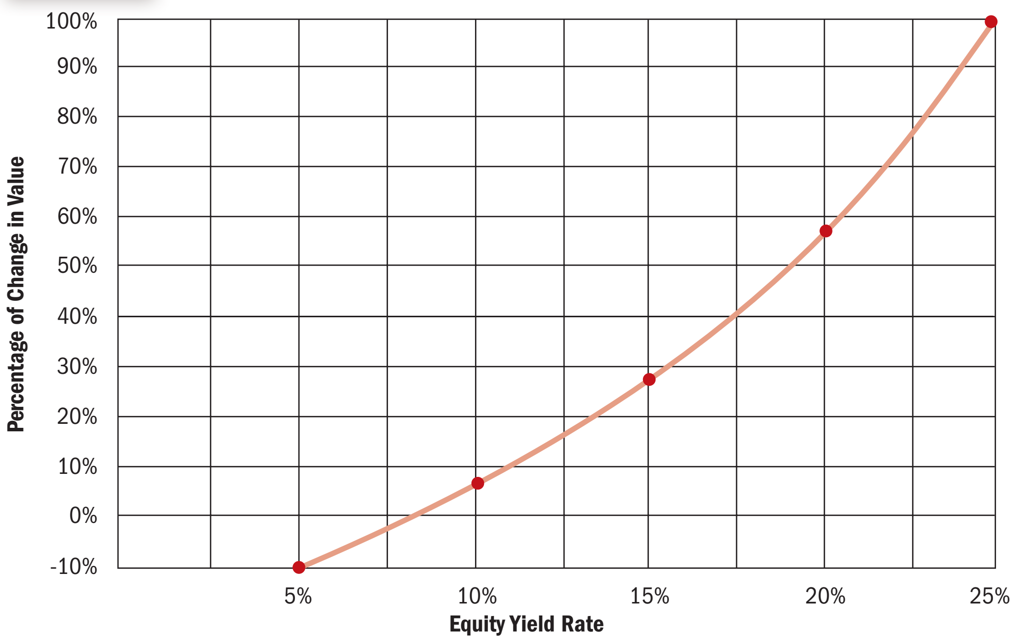


Figure C.3 Alternative Graphic Analysis



type of graphic analysis with the equity yield rate on the horizontal axis and the percentage change in value on the vertical axis. Graphs like these can be constructed manually by plotting three or more key points and connecting the points with a smooth curve. They can also be constructed using a computer.

The graph in Figure C.2 shows change in value and income under the *J*-factor premise with respect to time for equity yield rates of 5%, 10%, 15%, 20%, and 25%. It is assumed that $R_O = 0.11$, $I = 0.125$, $R_M = 0.135$, $M = 0.7$, and $\Delta_O = \Delta_I$.

The graph in Figure C.3 shows the change in value and income under the *J*-factor premise for equity yield rates ranging from 5% to 25% over a 10-year holding period. Again, it is assumed that $R_O = 0.11$, $I = 0.125$, $R_M = 0.135$, $M = 0.7$, and $\Delta_O = \Delta_I$.

After a graph is created, it must be interpreted by the appraiser. Usually the appraiser determines the range of property value changes (Δ_O) anticipated by the market and then forms an opinion as to the reasonableness of the overall capitalization rate. If the value changes are in line with the expectations of market participants and there is nothing unusual about the subject property, the overall rate being tested may be reasonable. If the value changes are not within the range expected by the marketplace, the overall capitalization rate should either be considered unreasonable and in need of further analysis or must be explained and accounted for.

Rate Extraction

Rate extraction is a technique that allows an appraiser to infer the market's expectation of yield and change in property value from a market-oriented overall capitalization rate. The key is to determine what assumptions about the yield rate and the change in property value are consistent with the overall capitalization rates derived from comparable sales. Although a specific yield rate or change in value cannot be identified using this approach, an analyst can determine what change in property value is needed to produce a given yield rate. That is, for each assumed yield rate, there is only one assumption about the change in property value that can be used with that rate to obtain the overall capitalization rate implied by comparable sales.

The following example illustrates this technique. The subject property is an apartment complex. Data on three comparable properties is given.

Factual Data on Three Apartment Complexes

	Sale 1	Sale 2	Sale 3
Number of units	240	48	148
Sale price	\$4,678,000	\$811,000	\$3,467,000
Cash down payment	\$1,300,000	\$462,145	\$1,370,000
Gross income	\$594,540	\$126,240	\$507,120
I_O	\$368,600	\$71,500	\$293,400

Comparative Factors	Sale 1	Sale 2	Sale 3
Price per unit	$19,492	$16,896	$23,426
Gross income per unit			
Annually	$2,477	$2,638	$3,426
Monthly	$206	$219	$285
Gross income multiplier (*GIM*)	7.870	6.420	6.830
Overall capitalization rate (R_O)	0.079	0.088	0.085
Loan-to-value ratio (*M*)	0.722	0.430	0.605
Mortgage constant (R_M)	0.107	0.127	0.136
Percent paid off (*P*)	-0.125	0.016	0.032
Equity capitalization rate (R_E)	0.006	0.059	0.006
Debt coverage ratio (*DCR*)	1.021	1.610	1.030

Using the mortgage-equity *J*-factor formula, pairs of Y_E and Δ_O can be extracted for each comparable sale. The formula for change in income and value is

$$\Delta_{O=I} = \frac{Y_E - M\,(Y_E + P \times 1/S_{\overline{n|}} - R_M) - R_O}{R_O\,J + 1/S_{\overline{n|}}}$$

The overall rate for each of the sales can be used in this formula to solve for the combinations of Y_E and Δ_O that would produce that overall capitalization rate. This data is shown in the following table:

Calculated Required Changes for the Three Sales (Five-Year Projection)

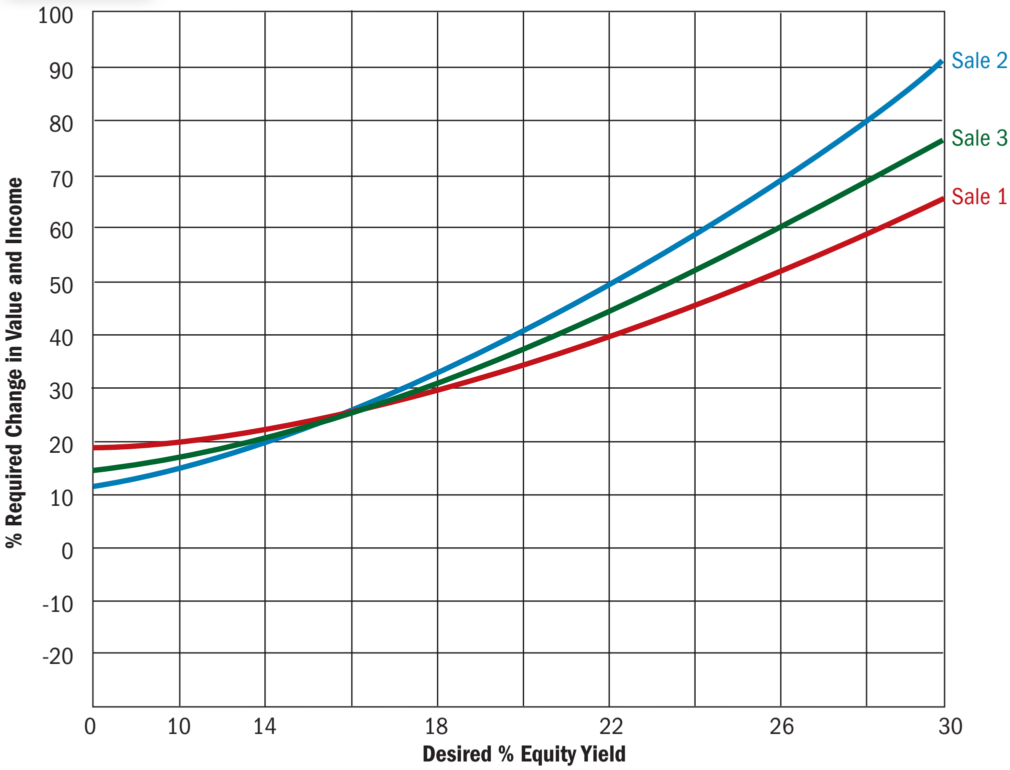
	% $\Delta_{O=I}$		
%Y_E	**Sale 1**	**Sale 2**	**Sale 3**
10	19.9	10.7	16.1
12	23.2	16.8	20.8
14	26.7	23.3	25.7
16	30.5	30.3	31.0
18	34.5	37.8	36.6
20	38.8	45.8	42.7
22	43.4	54.3	49.1
24	48.3	63.4	55.9
26	53.7	73.2	63.2
28	59.0	83.4	70.9
30	64.9	94.6	79.2

Note that the rate of change in property value is assumed to equal the rate of change in income. This reflects the appraiser's belief that this assumption is consistent with market perceptions. The relationship between equity yield and change in value and income can now be graphed (Figure C.4).

Once the graph is completed, the appraiser can draw certain conclusions. If the sales used accurately reflect market perceptions, every pair of equity yield rate and change in property value is a perfect pair. When the figures are inserted into the mortgage-equity formula to derive an overall capitalization rate, the resulting value estimate will be market-oriented.

In this case, any pair of Y_E and Δ_O that does not coincide with the lines on the graph is not market-oriented. The lines have different slopes and cross at some point

Figure C.4 Relationship Between Equity Yield and Total Change in Value and Income



because each sale has a different loan-to-value ratio (*M*). Furthermore, because of the differences in the loan-to-value ratios, some variation in the yield rate that equity investors would require for each of the sales would be expected. For example, Sale 1 had the highest loan-to-value ratio and, therefore, probably had the highest required yield rate because of its greater risk. The curves indicate reasonable assumptions about yield rates and changes in value that are consistent with the prices paid for comparable sales and the manner in which they were financed.

The graph can also be used to reflect the most likely pair of Y_E and Δ_O for developing an overall capitalization rate. By verifying current investor perceptions of the yield anticipated for the type of property being appraised, an appraiser can determine the necessary property value change. Then, with the mortgage-equity formula, the overall capitalization rate can be calculated. This overall rate will reflect typical investor assumptions for both yield and change in property value.

Compound Interest (Future Value of One)

This factor reflects the amount to which an investment or deposit will grow in a given number of time periods, including the accumulation of interest at the effective rate per period. It is also known as the *amount of one*.

$$S^n = (1 + i)^n$$

where: S^n = future value factor
i = effective rate of interest
n = number of compounding periods

and $S^n = (e)^{in}$ for continuous compounding

where: S^n = future value factor
i = nominal rate of interest
n = number of years
e = 2.718282

This factor is used to solve problems dealing with compound growth.

When money is invested or deposited at the beginning of a period in an account that bears interest at a fixed rate, it grows according to the interest rate and the number of compounding (conversion) periods that remains in the account. To illustrate how and why this growth occurs, consider an investment of $1.00, a nominal interest rate of 10% with annual compounding, and an investment holding period of five years.

Original investment	$1.00
Interest, first year at 10%	0.10
Accumulation, end of 1 year	$1.10
Interest, second year at 10%	0.11
Accumulation, end of 2 years	$1.21
Interest, third year at 10%	0.121
Accumulation, end of 3 years	$1.331
Interest, fourth year at 10%	0.1331
Accumulation, end of 4 years	$1.4641
Interest, fifth year at 10%	0.14641
Accumulation, end of 5 years	$1.61051

One dollar grows to $1.61051 in five years with interest at 10%, so the future value of one factor at 10% annually for five years is 1.610510; $1,000 would grow 1,000 times this amount to $1,610.51 over the same five years at the same 10% annual rate. When interest is not collected or withdrawn as it is earned, it is added to the capital amount and additional interest accumulates in subsequent periods. This process is called *compounding*.

The results of compounding can be calculated with the formula $(1 + i)^n$, where n is the number of compounding periods and i is the interest rate per period.

n		
1	$1.10 \times 1 = 1.10^1$	= 1.10
2	$1.10 \times 1.10 = 1.10^2$	= 1.21
3	$1.10 \times 1.10 \times 1.10 = 1.10^3$	= 1.331
4	$1.10 \times 1.10 \times 1.10 \times 1.10 = 1.10^4$	= 1.461
5	$1.10 \times 1.10 \times 1.10 \times 1.10 \times 1.10 = 1.10^5$	= 1.61051

Thus, the factors in Figure C.5—the amount of one or the future value of one—reflect the growth of $1.00 accumulating at interest for the number of compounding periods shown at the left and right sides of each page of tables. For example, the 10% annual column reveals a factor of 2.593742 for 10 periods. This means that $1.00 deposited at 10% interest compounded annually for 10 years will grow to $1.00 × 2.593742, or just over $2.59. In other words, $1.10^{10} = 2.593742$. The factors for seven and eight years

Figure C.5 Compound Interest Table for 10%

10% RATE i	1	2	3	4	5	6	10% RATE i
	AMOUNT OF $1	AMOUNT OF $1 PER PERIOD	SINKING FUND FACTOR	PRESENT WORTH OF $1	PRESENT WORTH OF $1 PER PERIOD	PARTIAL PAYMENT	
n	The amount to which $1 will grow with compound interest	The amount to which $1 per period will grow with compound interest	The amount per period which will grow with compound interest to $1	What $1 due in the future is worth today	What $1 payable periodically is worth today	The installment to repay $1 with interest	n
1	1.100 000	1.000 000	1.000 000	.909 091	.909 091	1.100 000	1
2	1.210 000	2.100 000	.476 190	.826 446	1.735 537	.576 190	2
3	1.331 000	3.310 000	.302 115	.751 315	2.486 852	.402 115	3
4	1.464 100	4.641 000	.215 471	.683 013	3.169 865	.315 471	4
5	1.610 510	6.105 100	.163 797	.620 921	3.790 787	.263 797	5
6	1.771 561	7.715 610	.129 607	.564 474	4.355 261	.229 607	6
7	1.948 717	9.487 171	.105 405	.513 158	4.868 419	.205 405	7
8	2.143 589	11.435 888	.087 444	.466 507	5.334 926	.187 444	8
9	2.357 948	13.579 477	.073 641	.424 098	5.759 024	.173 641	9
10	2.593 742	15.937 425	.062 745	.385 543	6.144 567	.162 745	10
11	2.853 117	18.531 167	.053 963	.350 494	6.495 061	.153 963	11
12	3.138 428	21.384 284	.046 763	.318 631	6.813 692	.146 763	12
13	3.452 271	24.522 712	.040 779	.289 664	7.103 356	.140 779	13
14	3.797 498	27.974 983	.035 746	.263 331	7.366 687	.135 746	14
15	4.177 248	31.772 482	.031 474	.239 392	7.606 080	.131 474	15
16	4.594 973	35.949 730	.027 817	.217 629	7.823 709	.127 817	16
17	5.054 470	40.544 703	.024 664	.197 845	8.021 553	.124 664	17
18	5.559 917	45.599 173	.021 930	.179 859	8.201 412	.121 930	18
19	6.115 909	51.159 090	.019 547	.163 508	8.364 920	.119 547	19
20	6.727 500	57.274 999	.017 460	.148 644	8.513 564	.117 460	20
21	7.400 250	64.002 499	.015 624	.135 131	8.648 694	.115 624	21
22	8.140 275	71.402 749	.014 005	.122 846	8.771 540	.114 005	22
23	8.954 302	79.543 024	.012 572	.111 678	8.883 218	.112 572	23
24	9.849 733	88.497 327	.011 300	.101 526	8.984 744	.111 300	24
25	10.834 706	98.347 059	.010 168	.092 296	9.077 040	.110 168	25
26	11.918 177	109.181 765	.009 159	.083 905	9.160 945	.109 159	26
27	13.109 994	121.099 942	.008 258	.076 278	9.237 223	.108 258	27
28	14.420 994	134.209 936	.007 451	.069 343	9.306 567	.107 451	28
29	15.863 093	148.630 930	.006 728	.063 039	9.369 606	.106 728	29
30	17.449 402	164.494 023	.006 079	.057 309	9.426 914	.106 079	30
31	19.194 342	181.943 425	.005 496	.052 099	9.479 013	.105 496	31
32	21.113 777	201.137 767	.004 972	.047 362	9.526 376	.104 972	32
33	23.225 154	222.251 544	.004 499	.043 057	9.569 432	.104 499	33
34	25.547 670	245.476 699	.004 074	.039 143	9.608 575	.104 074	34
35	28.102 437	271.024 368	.003 690	.035 584	9.644 159	.103 690	35
36	30.912 681	299.126 805	.003 343	.032 349	9.676 508	.103 343	36
37	34.003 949	330.039 486	.003 030	.029 408	9.705 917	.103 030	37
38	37.404 343	364.043 434	.002 747	.026 735	9.732 651	.102 747	38
39	41.144 778	401.447 778	.002 491	.024 304	9.756 956	.102 491	39
40	45.259 256	442.592 556	.002 259	.022 095	9.779 051	.102 259	40
41	49.785 181	487.851 811	.002 050	.020 086	9.799 137	.102 050	41
42	54.763 699	537.636 992	.001 860	.018 260	9.817 397	.101 860	42
43	60.240 069	592.400 692	.001 688	.016 600	9.833 998	.101 688	43
44	66.264 076	652.640 761	.001 532	.015 091	9.849 089	.101 532	44
45	72.890 484	718.904 837	.001 391	.013 719	9.862 808	.101 391	45
46	80.179 532	791.795 321	.001 263	.012 472	9.875 280	.101 263	46
47	88.197 485	871.974 853	.001 147	.011 338	9.886 618	.101 147	47
48	97.017 234	960.172 338	.001 041	.010 307	9.896 926	.101 041	48
49	106.718 957	1057.189 572	.000 946	.009 370	9.906 296	.100 946	49
50	117.390 853	1163.908 529	.000 859	.008 519	9.914 814	.100 859	50
51	129.129 938	1281.299 382	.000 780	.007 744	9.922 559	.100 780	51
52	142.042 932	1410.429 320	.000 709	.007 040	9.929 599	.100 709	52
53	156.247 225	1552.472 252	.000 644	.006 400	9.935 999	.100 644	53
54	171.871 948	1708.719 477	.000 585	.005 818	9.941 817	.100 585	54
55	189.059 142	1880.591 425	.000 532	.005 289	9.947 106	.100 532	55
56	207.965 057	2069.650 567	.000 483	.004 809	9.951 915	.100 483	56
57	228.761 562	2277.615 624	.000 439	.004 371	9.956 286	.100 439	57
58	251.637 719	2506.377 186	.000 399	.003 974	9.960 260	.100 399	58
59	276.801 490	2758.014 905	.000 363	.003 613	9.963 873	.100 363	59
60	304.481 640	3034.816 395	.000 330	.003 284	9.967 157	.100 330	60
n	$S^n = (1+i)^n$	$S_{\overline{n}\rceil} = \frac{S^n - 1}{i}$	$\frac{1}{S_{\overline{n}\rceil}} = \frac{i}{S^n - 1}$	$\frac{1}{S^n} = \frac{1}{(1+i)^n}$	$a_{\overline{n}\rceil} = \frac{1 - 1/S^n}{i}$	$\frac{1}{a_{\overline{n}\rceil}} = \frac{i}{1 - 1/S^n}$	n

$S = 1 + i$

indicate that $1.00 (or any investment earning 10% per year) will double in value in approximately 7.5 years. Similarly, an investment of $10,000 made 10 years ago, earning no periodic income during the 10-year holding period, must be liquidated in the current market at $10,000 × 2.593742, or $25,937.42, to realize a 10% return on the original investment.

This factor reflects the growth of the original deposit measured from the beginning deposit period. Thus, at the end of the first period at a rate of 10%, the original \$1.00 has grown to \$1.10 and the factor is 1.100000, as shown above.

Reversion Factors (Present Value of One)

This factor is the present value of \$1 (or other currency) to be collected at a given future time discounted at the effective interest rate for the number of periods between now and the date of collection. It is the reciprocal of the corresponding compound interest factor.

$$1/S^n = \frac{1}{(1+i)^n}$$

where: $1/S^n$ = present value factor
i = effective rate of interest
n = number of compounding periods

and

$$1/S^n = \frac{1}{(e)^{in}} \text{ for continuous compounding}$$

where: $1/S^n$ = present value factor
i = nominal rate of interest
n = number of years
$e = 2.718282$

This factor is used to solve problems that involve compound discounting.

As demonstrated in the discussion of future value, \$1.00 compounded annually at 10% will grow to \$1.610151 in five years. Accordingly, the amount that will grow to \$1.00 in five years is \$1.00 divided by 1.61051, or \$0.62092. In the 10% table, the present value of one factor for five years is 0.620921. In other words, \$1.00 to be collected five years from today has a present value of \$0.620921 when discounted at 10% per year. And \$10,000 to be collected five years from today, discounted at the same 10% annual rate, has a present value of \$10,000 × 0.620921, or \$6,209.21. The \$10,000 sum to be received in five years is a reversion.

Ordinary Level Annuity (Present Value of One per Period)

This factor represents the present value of a series of future installments or payments of \$1 (or other currency) per period for a given number of periods discounted at an effective interest rate. It is commonly referred to as the Inwood coefficient.

$$a_{\overline{n}|} = \frac{1 - 1/S^n}{i}$$

where: $a_{\overline{n}|}$ = level annuity factor
$1/S^n$ = present value factor
i = rate of interest yield

This factor is used in solving problems that deal with the compound discounting of cash flows that are level or effectively level.

Finding the present value of a future income stream is a discounting procedure in which future payments are treated as a series of reversions. The present value of a series of future receipts may be quickly ascertained using the precomputed present value of one per period factors for the selected discount rate provided the receipts are all equal in amount, equally spaced over time, and receivable at the end of each period.

If, for example, 10% per year is a fair rate of interest or discount, it would be justifiable to pay $0.909091 (i.e., the annual present value of $1 at 10%) for the right to receive $1.00 one year from today. Assuming that the cost of this right is $0.909091, the $1.00 received at the end of the year could be divided between principal and interest as follows.

Return of Principal	$0.90909
Interest on Principal for 1 Year @ 10%	0.09091
Total Received	$1.00000

If approximately $0.91 is the present value of the right to receive $1.00 of income one year from today at 10% interest, the present value of the right to receive $1.00 two years from today is less. According to the present value formula, the present value of $1.00 to be received two years from today is $0.826446. The present value of $1.00 payable at the end of two years can be confirmed with these calculations.

Return on Principal	$0.82645*
Interest for First Year at 10% on $0.82645	0.08264
	$0.90909
Interest for Second Year at 10% on $0.90909	0.09091
Total Principal Repayment + Interest Received	$1.00000

* Present value factor, 0.826446 × $1.00 = $0.82645 (rounded).

Similarly, the present value of the right to receive $1.00 at the end of three years is $0.751315. At the end of four years it is $0.683013, and at the end of the fifth year it is $0.620921. The present value of these rights to receive income at one-year intervals for five years is accumulated as the present value of $1.00 per year. This is known as the *compound interest valuation premise*, also referred to as the *ordinary annuity factor*. Therefore, the sum of the five individual rights to receive $1.00 each year, payable at the end of the year, for five years is $3.790787 (i.e., the 10% annual present value of one per period factor for five years).

Sum of Individual Present Values of $1.00 Payable at the End of the Period	
Present Value of $1.00 Due in 1 Year	$0.909091*
Present Value of $1.00 Due in 2 Years	0.826446*
Present Value of $1.00 Due in 3 Years	0.751315*
Present Value of $1.00 Due in 4 Years	0.683013*
Present Value of $1.00 Due in 5 Years	0.620921*
Total Present Value of $1.00 per Year for 5 Years	$3.790786**

* 10% present value of one factor.

** 10% present value of one per period factor is 3.790787; the difference is due to rounding.

The present value of one per period table for five annual discounting periods ($n = 5$) gives a factor that represents the total of the present values of a series of periodic amounts of $1.00, payable at the end of each period. The calculation presented above is unnecessary because multiplying $1.00 by the factor for the present value of $1 per year for five years produces the same present value ($1.00 × 3.790787 = $3.790787).

For appraisal purposes, the present value of one per period factor may be multiplied by a periodic income with the characteristics of an ordinary annuity to derive the present value of the right to receive that income stream. The future payments of income provide for recapture of, and interest on, this present value. Present value factors are multipliers and perform the same function as capitalization rates.

The 10% ordinary annuity factor for five years, 3.790787, represents the present value of each $1.00 of annual end-of-year collection based on a nominal annual discount rate of 10%. Tables and formulas for semiannual, quarterly, and monthly payments are also available. The ordinary annuity factor for semiannual payments in the 10% nominal annual rate table is 7.721735. If payment continues for five years, each $1.00 of semiannual payment represents $10.00 received but reflects only $7.72 of the discounted present value of monthly payments for five years. In the table for a 10% nominal rate, the monthly factor is 47.065369, indicating that the present value of an ordinary annuity income stream of 60 monthly payments of $1.00 each discounted at a nominal rate of 10% is 47.065369 × $1.00, or about $47.07.

Based on a 10% nominal rate, semiannual payments would involve an effective rate of 5%. In the 5% annuity table, the factor for 10 periods is 7.721735. This is the same factor shown in the 10% semiannual table for a five-year period. Thus, annuity factors for more frequent payment periods can be derived using nominal annual rate tables. Preprogrammed financial calculators can be used to facilitate these calculations.

In a calculation of the present value of an annuity income stream, it may be desirable to assume that periodic payments are made at the beginning rather than the end, of each payment period. The present value of an annuity payable in advance is equal to the present value of an ordinary annuity in arrears multiplied by the base (i.e., 1 plus the effective interest rate for the discounting period: $1 + i$). Thus, the present value of semiannual payments in advance over a five-year period discounted at a nominal rate of 10% becomes $1.00 × 7.721735 × 1.05 = $8.107822, or $8.11, compared to $7.72 as computed for payments received at the end of each payment period.

Ordinary Annuities Changing in Constant Amounts

Present Value of Annual Payments Starting at One and Changing in Constant Amounts

$$PVF = (1 + h\,n)\,a_{\overline{n}|} - \frac{h\,(n - a_{\overline{n}|})}{i}$$

where:

PVF = present value factor

h = annual increase or decrease after first year*

n = number of years

$a_{\overline{n}|}$ = PVF for ordinary level annuity

i = rate of interest yield

* h is positive for an increase and negative for a decrease

This factor is used to solve problems dealing with the compound discounting of cash flows that are best represented by a straight-line pattern of change.

This factor is similar to the ordinary level annuity table, but the annual receipts are converted into constant dollar amounts. For instance, assume that the amount to be received one year from today is $10,000, additional future receipts are expected to increase $1,000 per year for the next nine years, and 15% per year is a fair rate of interest. According to the 15% annual present value of one factor, it would be justifiable to pay $67,167 for the right to receive $10,000 one year from today and nine additional payments growing at $1,000 per year for nine additional years. The table for 15% indicates that the factor to be applied to the initial receipt is 6.7167.

Year	Income	×	Present Value Factor	=	Present Value
1	$10,000	×	0.869565	=	$8,695.65
2	11,000	×	0.756144	=	8,317.58
3	12,000	×	0.657516	=	7,890.19
4	13,000	×	0.571753	=	7,432.79
5	14,000	×	0.497177	=	6,960.48
6	15,000	×	0.432328	=	6,484.92
7	16,000	×	0.375937	=	6,014.99
8	17,000	×	0.326902	=	5,557.33
9	18,000	×	0.284262	=	5,116.72
10	19,000	×	0.247185	=	4,696.52
Present value					$67,167.17

$$\frac{\text{Present Value}}{\text{Initial Receipt}} = \text{Factor}$$

$$\frac{\$67{,}167.17}{\$10{,}000.00} = 6.7167$$

Ordinary Annuities Changing in Constant Ratio

Present Value of Annual Payments Starting at One and Changing in Constant Ratio

$$PVF = \frac{1 - \left(\frac{1+x}{1+i}\right)^n}{i - x}$$

where:

PVF = present value factor

x^* = constant ratio change in income

n = number of years

i = rate of interest or yield

* x is positive for an increase and negative for a decrease

This factor is used to solve problems dealing with the compound discounting of cash flows that are best represented by an exponential-curve pattern of change.

Sinking Fund Factors

Periodic Payment to Grow to One

This factor represents the level periodic investment or deposit required to accumulate to $1 (or other unit of currency) in a given number of periods including interest at the effective rate. It is commonly known as the *amortization rate* and is the reciprocal of the corresponding sinking fund accumulation factor.

$$1/S_{\overline{n|}} = \frac{i}{S^n - 1}$$

where:

$1/S_{\overline{n|}}$ = sinking fund factor

i = effective rate of interest

n = number of compounding periods

S^n = future value factor

This factor is used to solve problems that involve calculating required sinking fund deposits or providing for the change in capital value in investment situations where the income or payments are level.

When deposits are made at the end of each compounding period, sinking fund factors reflect the fractional portion of $1.00 that must be deposited periodically at a specified interest rate to accumulate to $1.00 by the end of the series of deposits.

If $10,000 is to be accumulated over a 10-year period and annual deposits are compounded at 10% interest, the factor shown on the 10-year line of the annual column in the 10% sinking fund table indicates that each annual deposit must amount to $10,000 × 0.062745, or $627.45.

Sinking Fund Accumulation Factors

Future Value of Periodic Payments of One

The future value of periodic payments of one factor represents the total accumulation of principal and interest on a series of deposits or installments of $1 (or other currency) per period for a given number of periods with interest at the effective rate per period. It is also known as the *amount of one per period*. It is the reciprocal of the corresponding sinking fund factor.

$$S_{\overline{n}|} = \frac{S^n - 1}{i}$$

where:

$S_{\overline{n}|}$ = sinking fund accumulation factor

i = effective rate of interest

S^n = future value factor

This factor is used to solve problems that involve the growth of sinking funds or the calculation of capital recovery in investment situations where the income or payments are level.

Sinking fund accumulation factors are similar to the future value of one (amount of one) factors except that deposits are periodic (in a series) and are assumed to be made at the end of the first compounding period and at the end of each period thereafter. Thus, the initial deposit, which is made at the end of the first period, has earned no interest and the factor for this period is 1.000000.

If compounding at 10% per year for 10 years is assumed, a factor of 15.937425 reveals that a series of 10 deposits of $1.00 each made at the end of each year for 10 years will accumulate to $1.00 × 15.937425, or almost $15.94.

Direct Reduction Loan Factors

Monthly Payment and Annual Constant per One of Loan

Payment: $$1/a_{\overline{n}|} = \frac{i}{1 - 1/S^n}$$

Annual constant: $$R_M = 12/a_{\overline{n}|}$$

where:

$1/a_{\overline{n}|}$ = direct reduction loan factor

$1/S^n$ = present value factor

i = effective rate of interest

R_M = annual constant

Part paid off: $P = \dfrac{R_M - 12i}{R_{M_p} - 12i}$

where: R_M = actual annual constant

R_{M_p} = annual constant for projection period

i = effective rate of interest

This factor is used to solve problems dealing with monthly payment, direct reduction loans. Payments and constants for quarterly, semiannual, and annual payment loans can be obtained by calculating the reciprocals of the present value of one per period factors.

These factors, which are known as *mortgage constants for loan amortization*, reflect the amount of ordinary annuity payment that $1.00 will purchase. They indicate the periodic payment that will extinguish the debt and pay interest on the declining balance of the debt over the life of the payments. The mortgage constant may be expressed in terms of the periodic payments. A mortgage constant related to a monthly payment is the ratio of the monthly payment amount to the original amount of the loan. Whether payments are monthly, semiannual, or annual, the mortgage constant is usually expressed in terms of the total payments in one year as a percentage of the original loan amount. This is called the *annual constant* and is represented by the symbol R_M. As the loan is paid off and the outstanding balance is reduced, a new annual mortgage constant can be calculated as the ratio of total annual payments to the unpaid balance of the loan at that time.

A loan of $10,000 to be amortized in 10 annual end-of-year payments at a mortgage interest rate of 10% would require level annual payments of $10,000 × 0.162745, the 10% direct reduction annual factor for 10 years. If monthly payments were made at 10% over 10 years, the amount of each payment would be $132.15 (i.e., $10,000 × 0.013215). The annual mortgage constant in this case would be 0.158580, or 12 × 0.013215.

Direct reduction factors consist of the interest rate plus the sinking fund factor at the specific point in time. They are reciprocals of the corresponding ordinary level-annuity factors.

Interrelationships Among the Factors

Note that mathematical relationships exist among the formulas for the various factors. These relationships can be useful in understanding the factors and solving appraisal problems. For example, appraisers should know that the factors in the ordinary level annuity and direct reduction loan tables are reciprocals; the factors in the ordinary level annuity table can be used as multipliers instead of using the direct reduction loan factors as divisors.

Reciprocals

The factors in some of the tables are reciprocals of those in other tables. This is indicated by their formulas.

Future Value of One and Reversion Factors

$$S^n \text{ and } \frac{1}{S^n}$$

The reversion factor at 12% for 10 years with annual compounding is 0.321973, which is the reciprocal of the future value of one factor.

$$0.321973 = 1/3.105848$$

Sinking Fund Accumulations and Sinking Fund Factors

$$S_{\overline{n}|} \text{ and } 1/S_{\overline{n}|}$$

The sinking fund factor at 12% for 10 years with annual compounding is 0.056984, which is the reciprocal of the sinking fund accumulation factor.

$$0.056984 = 1/17.548735$$

Ordinary Level Annuity and Direct Reduction Loan Factors

$$a_{\overline{n}|} \text{ and } 1/a_{\overline{n}|}$$

The direct reduction loan factor at 12% for 10 years with annual compounding is 0.176984, which is the reciprocal of the ordinary level annuity factor.

$$0.056984 = 1/17.548735$$
$$0.176984 = 1/5.650223$$

Summations

Ordinary Level Annuity Factors

An ordinary level annuity factor represents the sum of the reversion factors for all periods up to and including the period being considered. For example, the ordinary level annuity factor for five years at 12% with annual compounding is 3.604776, which is the sum of all the reversion factors for Years 1 through 5.

$$\begin{array}{r} 0.892857 \\ 0.797194 \\ 0.711780 \\ 0.635518 \\ \underline{0.567427} \\ 3.604776 \end{array}$$

Direct Reduction Loan Factors

A direct reduction loan factor represents the sum of the interest, yield, or discount rate stated at the top of the table and the sinking fund factor. For example, the direct reduction loan factor at 12% for 10 years with monthly compounding is 0.1721651, which is the sum of 0.12 plus the monthly sinking fund factor of 0.0043471 times 12 (0.12 + 0.0521651 = 0.1721651).

Conversely, the sinking fund factor can be obtained by subtracting the interest rate from the direct reduction loan factor. The sinking fund factor at 12% for 10 years with monthly compounding is 0.1721651 – 0.12 = 0.0521651. In addition, the interest rate can be obtained by subtracting the sinking fund factor from the direct reduction loan factor. Given a mortgage constant of 0.1721651 with monthly compounding for 10 years, the interest rate is 0.1721651 – 0.0521651 = 0.12000, or 12.0%.